

The role of a form of vector potential — normalization of the antisymmetric gauge

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Results obtained for the antisymmetric gauge $\mathbf{A} = [Hy, -Hx]/2$ by Brown and Zak are compared with those based on pure group-theoretical considerations and corresponding to the Landau gauge $\mathbf{A} = [0, Hx]$. Imposing the periodic boundary conditions one has to be very careful since the first gauge leads to a factor system which is not normalized. A period N introduced in Brown's and Zak's papers should be considered as a *magnetic* one, whereas the *crystal* period is in fact $2N$. The 'normalization' procedure proposed here shows the equivalence of Brown's, Zak's, and other approaches. It also indicates the importance of the concept of *magnetic cells*. Moreover, it is shown that factor systems (of projective representations and central extensions) are gauge-dependent, whereas a commutator of two magnetic translations is gauge-independent. This result indicates that a *form* of the vector potential (a gauge) is also important in physical investigations.

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I. INTRODUCTION

The discovery of the quantum Hall effect^{1,2} led to remarkable interest in two-dimensional electron systems subjected to a magnetic field.³ Since 1980 authors working in different fields — from applied to mathematical physics — have considered related problems and many new features have been observed and discussed.⁴ One of the most interesting questions is the dynamic of two-dimensional electrons in a periodic potential and an external magnetic field.⁵ The first results, in the tight binding approximation, were presented by Peierls,⁶ shortly after Landau's⁷ discovery of the quantization of electron states in a magnetic field. A new impact was due to Brown⁸ and Zak^{9,10} who independently introduced magnetic translation operators in two different, but equivalent, ways. Both approaches were based on group-theoretical considerations and led to the broadening of the Landau levels and quantization of a magnetic field.^{8,11} Although more than thirty years have passed, their papers are still considered as fundamental ones.⁵ Brown and Zak proved that the problem considered is in fact two-dimensional and their investigations confirmed the importance of projective representations and central extensions in quantum physics.¹² On the other hand, Zak's and Brown's results were not gauge-independent — only a completely antisymmetric vector potential was considered by both authors. An attempt to consider gauge-equivalent vector potentials leads to some ambiguities and misconceptions if it is not done carefully. A bit simpler and more clear results can be obtained from pure group-theoretical considerations. For example, Divakaran and Rajagopal did not consider gauges at all and they worked with central extensions and projective representations only.¹³ However, pure mathematical description may not provide us with an intuitive image of the physical phenomena. Moreover, many experiments and theories indicate the importance of vector potential,¹⁴ so it is necessary to include gauges and potentials in considerations.

The aim of this paper is to show sources of misconceptions, ambiguities, and unexpected gauge-dependence of the problem. In particular, factor systems of projective representations and central extensions introduced by Brown and Zak have been carefully checked and compared with those obtained from pure group-theoretical considerations.^{13,15,16} It occurs that they can be considered as standard but they are not normalized.^{12,17} This last fact is the main source of differences between Brown's and Zak's approaches. Moreover, it indicates points at which a form of the vector potential is important, *i.e.*, the points at which the problem is not gauge-independent.

In this paper we propose a procedure of 'normalization' of those factor systems, which enables us to identify irreps introduced by Brown and Zak. A comparison of these irreps with those obtained for central extensions of finite translation groups leads to a concept of the so-called *magnetic cells*¹⁰ and shows that Brown and Zak considered in fact finite lattices with a period $2N$ not N .

For the sake of clarity, the following simplifications arising from the quoted papers are assumed. Position (\mathbf{r} , \mathbf{R}), momentum (\mathbf{p}), and vector potential (\mathbf{A}) are considered to be two-dimensional vectors. Note that $\mathbf{r} = (x, y)$ is any vector of \mathbb{R}^2 , whereas $\mathbf{R} = (X, Y) \in \mathbb{Z}^2$ denotes a vector of a square lattice with $\mathbf{a}_1 = \hat{\mathbf{x}}$ and $\mathbf{a}_2 = \hat{\mathbf{y}}$, so the area of the elementary cell is equal to 1. The magnetic field is perpendicular to the x - y plane and $\mathbf{H} = H\hat{\mathbf{z}}$. The periodic boundary conditions are imposed on representations of \mathbb{Z}^2 and the periods are equal, *i.e.*, $N_1 = N_2 = N$; the finite translation group and its representations can be considered equivalently.

The paper is organized as follows. In Sec. II the most fundamental formulas of Brown's and Zak's papers are recalled and equivalence of their approaches are indicated. Basic properties of projective representations are briefly

presented, too. The role of factor systems is briefly discussed in Sec. III. The next section is devoted to determination of the equivalence of different approaches. From the physical point of view it is done by introducing the concept of magnetic cells. The results obtained are discussed in Sec. V.

II. DIFFERENT DESCRIPTIONS OF MAGNETIC TRANSLATION GROUPS

From the algebraic point of view there are two equivalent descriptions of the magnetic translation operators. Brown⁸ investigated a *projective* representation of the translation group T then imposed the magnetically periodic boundary conditions on it. On the other hand, Zak⁹ introduced a closed set of noncommuting operators which, in fact, form a covering group T' of T so its standard (vector) representations are *projective* representations of T .¹⁷ The finiteness of these representations was again achieved by imposing the periodic boundary conditions. These two approaches are related by a formula which follows from the induction procedure if one constructs representations of the covering group. Since T' is a central extension of T by the group $U(1)$ (or its subgroup referred to hereafter as a group of factors and denoted F)^{15,16} then its (vector) representations can be written as

$$\Xi[\alpha, \mathbf{R}] = \Gamma(\alpha)D(\mathbf{R}), \quad (1)$$

where $\alpha \in F \subset U(1)$, $\mathbf{R} \in T$, Γ is a vector representation of F and D is a *projective* representation of T . A factor system $m(\mathbf{R}, \mathbf{R}')$ of this representations is determined by the relation¹⁷

$$D(\mathbf{R})D(\mathbf{R}') = m(\mathbf{R}, \mathbf{R}')D(\mathbf{R} + \mathbf{R}'), \quad (2)$$

whereas the multiplication rule for T' reads

$$[\alpha, \mathbf{R}][\alpha', \mathbf{R}'] = [\alpha\alpha'\mu(\mathbf{R}, \mathbf{R}'), \mathbf{R} + \mathbf{R}'] \quad (3)$$

with $\mu(\mathbf{R}, \mathbf{R}')$ being a factor system of a central extension. These factor systems are related to each other by the formula

$$m(\mathbf{R}, \mathbf{R}') = \Gamma[\mu(\mathbf{R}, \mathbf{R}')] . \quad (4)$$

This relation establishes the equivalence of both approaches. Moreover, both authors assumed the antisymmetric vector potential (gauge) $\mathbf{A} = (\mathbf{H} \times \mathbf{r})/2 = [Hy, -Hx]/2$ and were not able to generalize their considerations to other gauges, in particular their approaches did not include the Landau gauge. On the other hand, their results and some conclusions are different in some points which will be discussed here¹⁸ and compared with the results obtained for the Landau gauge.

All considerations and formulas given above are also valid for a finite group T_N and its (finite-dimensional) representations. In addition, we can apply to this case a version of the Burnside theorem which reads that nonequivalent irreducible projective representations of T_N with the same factor system $m(\mathbf{R}, \mathbf{R}')$ satisfy the following condition¹⁷

$$\sum_j [{}^jD]^2 = |T_N| = N^2, \quad (5)$$

where j labels nonequivalent representations (there is no expression for a number of these representations) and $[{}^jD]$ denotes the dimension of jD . Since F is an Abelian group then it has $|F|$ irreducible nonequivalent representations and each of them determines different (nonequivalent) factor system $m(\mathbf{R}, \mathbf{R}')$ according to (4). It follows from (5) that irreducible representations of T'_N determined by (1) satisfy the Burnside theorem.

Brown⁸ defined a magnetic translation operator as

$$\hat{T}(\mathbf{R}) = \exp[-i\mathbf{R} \cdot (\mathbf{p} - e\mathbf{A}/c)/\hbar], \quad (6)$$

where \mathbf{p} is the kinetic momentum and \mathbf{A} is the vector potential such that $\nabla \times \mathbf{A} = \mathbf{H}$. These operators form a projective representation of T with a factor system⁸

$$m(\mathbf{R}, \mathbf{R}') = \exp[-\pi i(\mathbf{R} \times \mathbf{R}') \cdot \mathbf{H}/\varphi_0] \quad (7)$$

where $\varphi_0 = ch/e$. Brown showed that one can impose the periodic boundary conditions $\hat{T}(N\mathbf{a}_j)\psi = \psi$ if (notice simplifications assumed in this paper — in fact, H denotes hereafter the magnetic flux through one primitive cell)

$$H = \frac{l}{N}\varphi_0, \quad (8)$$

where l is mutually prime with N , *i.e.* $\gcd(l, N) = 1$. Hence a factor system of a finite projective representation lD for H satisfying (8) is given as

$$m_l([X, Y], [X', Y']) = \exp[-\pi i l (XY' - YX')/N]. \quad (9)$$

Brown showed that there is the unique (up to equivalence) irreducible projective representation with a dimension N and matrix elements⁸

$${}^lD_{jk}[X, Y] = \exp\left[\pi i \frac{lX}{N}(Y + 2j)\right] \delta_{j, k-Y}, \quad (10)$$

where $j, k = 0, 1, \dots, N-1$ and $\delta_{j, k}$ is calculated modulo N (Brown labeled rows and columns by $j, k = 1, 2, \dots, N$). Zak⁹ considered a covering group of the translation group consisting of operators

$$\tau(\mathbf{R}|\mathbf{R}_1, \dots, \mathbf{R}_j) = \widehat{T}(\mathbf{R}) \exp[2\pi i \phi(\mathbf{R}_1, \dots, \mathbf{R}_n)/\varphi_0], \quad (11)$$

where $\sum_i \mathbf{R}_i = \mathbf{R}$ and $\phi(\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_j)$ is the flux of the magnetic field through a polygon enclosed by a loop consisting of the vectors $\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_j, -\mathbf{R}$. The periodicity condition was the same as (8) for even N but for odd N Zak proved that the condition

$$H = 2\frac{l}{N}\varphi_0 \quad (12)$$

should be satisfied. This condition implies that for even N the number of different factors is $2N$, whereas it equals N for odd N .^{9,10} The result obtained by Zak agreed with Azbel's considerations¹⁹ who showed that wave functions had to be periodic functions of H with the period $2\varphi_0$. It is worthwhile noting that Azbel also worked with the antisymmetric gauge. Zak did not introduce a factor system in an explicit way (it was not necessary in his constructions of representations) but it can be easily found by considering multiplication of coset representatives $\tau(\mathbf{R}|\mathbf{R})$,^{17,20} which simply are equal to $\widehat{T}(\mathbf{R})$ [see (11)]. Therefore the factor system is also given by (9), but now it is the factor system of the covering group being a central extension so it should be denoted as $\mu_l(\mathbf{R}, \mathbf{R}')$. Zak¹⁰ also showed that matrix elements of an irreducible N -dimensional representation should be (only the coset representatives $\tau(\mathbf{R}, \mathbf{R})$ are taken into account here)²¹

$${}^lD_{jk}\left[\tau([X, Y]||[X, Y])\right] = \exp\left[2\pi i \frac{lX}{bN}(Y + 2k)\right] \delta_{j, k+Y}, \quad (13)$$

where $b = 1, 2$ for N odd and even, respectively. It is obvious that this representation corresponds to the irreducible representation $\Gamma(\alpha) = \alpha$ of the factor group F , so $m_l(\mathbf{R}, \mathbf{R}') = \mu_l(\mathbf{R}, \mathbf{R}')$. According to (8) and (12) changes of H are related to changes of l but they were interpreted in different ways. In Brown's considerations H determines a factor system of projective representations in a direct way — different values of H satisfying (8) lead to nonequivalent projective representations. On the contrary, Zak considered different (nonequivalent) central extensions of T with factor systems μ_l . However, Zak assumed that only the representations $\Gamma(\alpha) = \alpha$ were physical whereas the others were rejected as nonphysical in further considerations.^{10,22} It means, according to Zak, that for a central extension with a factor system μ_l one has to find projective representations lD with a factor system $m_l = \Gamma(\mu_l) = \mu_l$. The same result can be obtained while considering only the factor system μ_1 and next all irreducible representations Γ_l of F such that $\Gamma_l(\mu_1) = m_l$. Thus all representations necessary in physical applications, considered by Zak as representations of different although isomorphic groups, can be obtained by use of 'nonphysical' representations Γ_l with $l > 1$. Nevertheless, it seems that the representations introduced by Brown (10) could be used in Zak's approach to construct (vector) representations of T' (finite or not) according to (1). A comparison of (10) and (13) shows that for odd N Brown and Zak used different representations. However, for even N ($b = 2$) we have

$${}^lD_{jk}[X, -Y] = {}^lD_{jk}\left[\tau([X, Y]||[X, Y])\right], \quad (14)$$

where the sign ' $-$ ' originates from a different choice of the sign of e assumed by Zak¹⁸ (in Zak's approach eigenvectors of $D[1, 0]$ are permuted by $D[0, 1]$ in the opposite direction than that assumed in Brown's definition).

The third approach is based on pure group-theoretical considerations and consists in determination of all possible central extensions of a finite group \mathbb{Z}_N^2 (in general $\mathbb{Z}_{N_1} \otimes \mathbb{Z}_{N_2}$) by an infinite ($U(1)$) or finite ($C_N = \{\alpha \in \mathbb{C} \mid \alpha^N = 1\}$)

group of factors F .^{13,15,16} It was shown, by means of the Mac Lane method, that all nonequivalent factor systems corresponding to finite magnetic translation groups can be written as^{15,16}

$$\mu_k([X, Y], [X', Y']) = \exp(2\pi i k Y X' / N) \quad (15)$$

with $k = 0, 1, \dots, N - 1$. Some important facts have to be mentioned:

- This formula resembles the Landau gauge $\mathbf{A} = [0, Hx]$; recently it has been shown that this convergence is not accidental.²³
- The fraction k/N can be interpreted as H/φ_0 ,^{15,16} so the resulting numbers constitute a periodic function of H with the period φ_0 in agreement with Brown's result but contrary to Zak's and Azbel's results.
- In both Brown's and Zak's approaches a (group-theoretical) commutator of two magnetic translations, corresponding to vectors $[X, Y]$ and $[X', Y']$, is equal to

$$c([X, Y], [X', Y']) = \exp[-2\pi i (XY' - YX')H/\varphi_0] \quad (16)$$

and it is the same as obtained in the cited papers^{15,16} if H does not satisfy (12) but (8).

- There are no additional conditions imposed on k and on N (i.e., the results are valid for both odd and even N and for $\gcd(k, N) > 1$).

Taking into account only parameters $k = l$ mutually prime with N it can be shown that N -dimensional irreducible projective representations of T_N (or 'physical' vector representations of the extension of T_N by C_N) have the following matrix elements

$${}_3^l D_{jk}[X, Y] = \exp\left(2\pi i \frac{l}{N} Xj\right) \delta_{j, k-Y}. \quad (17)$$

Representations with $\gcd(k, N) > 1$ were briefly discussed elsewhere,²² but the difference between odd and even N was not considered there.

III. PROJECTIVE REPRESENTATIONS — STANDARD AND NORMALIZED FACTOR SYSTEMS

To compare different descriptions of the magnetic translation groups we have to discuss not only projective representations themselves but also their factor systems. To begin with we recall now some definitions related to factor systems and their properties.^{17,20,24,25} As one can see factor systems appear in the definition of a projective representation (2) and in the multiplication rule for a central extension of groups (3). Factor systems m_l are determined directly (as in Brown's approach) or via factor systems μ_l for central extensions by means of the 'physical' representations $\Gamma(\alpha) = \alpha$ and the formula (4).

A factor system $m: T \times T \rightarrow \mathbb{C}$ has to satisfy the following condition^{20,25}

$$m(\mathbf{R}, \mathbf{R}')m(\mathbf{R} + \mathbf{R}', \mathbf{R}'') = m(\mathbf{R}', \mathbf{R}'')m(\mathbf{R}, \mathbf{R} + \mathbf{R}'') \quad (18)$$

for all $\mathbf{R}, \mathbf{R}', \mathbf{R}''$. A *trivial* factor system $t(\mathbf{R}, \mathbf{R}')$ is determined by any mapping $f: T \rightarrow \mathbb{C}$ according to

$$t(\mathbf{R}, \mathbf{R}') = f(\mathbf{R})f(\mathbf{R}')/f(\mathbf{R} + \mathbf{R}'). \quad (19)$$

Since T is Abelian then each trivial factor system is symmetric, i.e., $t(\mathbf{R}, \mathbf{R}') = t(\mathbf{R}', \mathbf{R})$. If

$$m'(\mathbf{R}, \mathbf{R}') = t(\mathbf{R}, \mathbf{R}')m(\mathbf{R}, \mathbf{R}') \quad (20)$$

then factor systems m and m' are called *equivalent*. Notice, however, that the projective representations determined by equivalent factor systems are *nonequivalent*.¹⁷ Since all factor systems for a given T form an Abelian group Φ and a set of trivial factor systems Θ is its normal subgroup then elements of the factor group $M = \Phi/\Theta$ (known as the *Schur multiplier*) correspond to representatives of classes of equivalent factor systems. A factor system is called *standard* if it satisfies

$$m(\mathbf{R}, \mathbf{0}) = m(\mathbf{0}, \mathbf{R}) = 1, \quad \forall \mathbf{R}. \quad (21)$$

A factor system of an N -dimensional projective representation is *normalized* if

$$m(\mathbf{R}, \mathbf{R}') \in C_N, \quad \forall \mathbf{R}, \mathbf{R}' \quad (22)$$

(i.e., each factor is the N th root of 1).

It is well known that the Schur multiplier of \mathbb{Z}_N^2 is C_N ,²⁶ so the factor system (15) is normalized and standard since $m_k([0, 0], [X, Y]) = 1$. Moreover, it is periodic with respect to Y and X' — the period is equal to N . In particular we have $m_k([N, 0], [X, Y]) = m_k([X, Y], [N, N]) = 1$, etc. On the other hand, the factor system (9) of N -dimensional representations (10) or (13) is not normalized because some of factors do not belong to C_N but to C_{2N} instead. It also means that this system is standard, because $m_l([0, 0], [X, Y]) = 1$, but it appears that N does not serve as a period because, for example, $m_l([0, N], [1, 0]) = \exp(\pi i l) = (-1)^l$. This fact stirs up a conflict between the conditions obtained by Brown and those obtained by Zak and, moreover, leads to difficulties in studying magnetic translations for the antisymmetric gauge. Of course, one may work with factor systems (and, hence, representations) which are neither standard nor normalized, but such considerations have to be carried very carefully and results obtained have to be carefully interpreted, too.¹⁷ Brown and Zak did not check normalization of their factor systems and this led to ambiguity of their results [cf. (8) and (12)].

At first let us notice that Brown took into account one requirement only, namely⁸

$$\hat{T}(N\mathbf{a}_j)\hat{T}(\mathbf{R}) = \hat{T}(\mathbf{R})\hat{T}(N\mathbf{a}_j), \quad (23)$$

i.e., that $\hat{T}(N\mathbf{a}_j)$ would commute with any other operator. On the other hand, Zak demanded in addition that⁹

$$m(N\mathbf{a}_j, N\mathbf{a}_k) = 1, \quad (24)$$

i.e., that ${}^lD(N\mathbf{a}_j)$ should behave as a constant factor. As follows from (9)

$$m_l([N, 0], [0, N]) = (-1)^{lN},$$

so for odd N the magnetic field H has to be twice as high as in (8). Note that representations (17), corresponding to the Landau gauge, satisfy, for both odd and even N , the following stronger condition

$${}^l_3D(N\mathbf{a}_j){}^l_3D(\mathbf{R}) = {}^l_3D(\mathbf{R} + N\mathbf{a}_j) = {}^l_3D(\mathbf{R}), \quad (25)$$

i.e., both $N\mathbf{a}_j$ and $\hat{T}(N\mathbf{a}_j)$ are equal to the unit element in the translation group T_N and in the group of magnetic translation operators, respectively. The condition (12) (for odd N) removes these problems but, however, leads to another question why odd and even N should be considered separately while both cases can be evidently treated as one with the use of the Landau gauge.

While considering restrictions imposed on H by the periodic boundary conditions with the Landau gauge, i.e., the standard factor system (15), one can see that the condition $H = k\varphi_0/N$ is sufficient. So, it seems that Brown's approach is well-supported. Moreover, it should be noted that Zak weakened his requirements and later on he considered only Brown's condition.²⁷ To enlighten the problem we have to check whether the factor systems (9) and (15) are equivalent or not. At first note that the group-theoretical commutator of operators of any projective representations (of an Abelian group T) is equal to

$$c(\mathbf{R}, \mathbf{R}') = D(\mathbf{R})D(\mathbf{R}')[D(\mathbf{R}')D(\mathbf{R})]^{-1} = \frac{m(\mathbf{R}, \mathbf{R}')}{m(\mathbf{R}', \mathbf{R})}, \quad (26)$$

then it is the same for all equivalent factor systems. Since the equivalence of factor systems means the equivalence of vector potentials (gauges),^{2,23} then the above commutator does not depend on \mathbf{A} but rather on \mathbf{H} and in this sense this commutator only (not a factor system) has the physical meaning — if $D(\mathbf{R})$ represents a lattice translation in the presence of a magnetic field then the commutator corresponds to a loop determined by vectors $\mathbf{R}, \mathbf{R}', -\mathbf{R}, -\mathbf{R}'$ and its value depends on the flux through a nonprimitive, in general, cell determined by these vectors. So, Brown's requirement (23) leading to the condition (8) was based on a reasonable assumption. However, the factor system considered was not normalized which led to a disagreement with Zak's results.

IV. EQUIVALENCE OF FACTOR SYSTEMS AND REPRESENTATIONS

Let us consider a mapping $f_w[X, Y] = \exp(2\pi i wXY)$, $w \in \mathbb{R}$, which determines the following trivial factor system

$$t_w([X, Y], [X', Y']) = \exp[-2\pi i w(XY' + YX')]. \quad (27)$$

The factors obtained belong to C_N , *i.e.*, t is standard and normalized, if $w = j/N$. For example for $j = k$ the factor system (15) is transformed to

$$\bar{\mu}_k([X, Y], [X', Y']) = (\mu_k \circ t_{k/N})([X, Y], [X', Y']) = \exp(-2\pi i kXY'/N), \quad (28)$$

which corresponds to another form of the Landau gauge $\bar{\mathbf{A}} = [-Hy, 0]$. It is important that if t is not normalized then a new factor system $m' = tm$ is not normalized, too. This is, however, the case which leads to the factor system (9) determined by Brown and Zak — one has to put $w = k/2N$. This, and the previous discussion on the commutator, proves that the stronger condition introduced by Zak following from (24) is superfluous. It can be easily shown for odd N , since for l mutually prime with odd N also $\gcd(2l, N) = 1$ (the mapping $l \mapsto 2l$ is an automorphism of \mathbb{Z}_N which changes the order of elements only). Therefore in the formulas obtained by Brown, (8)–(10), one can replace $l < N$ in the following way

$$l = \begin{cases} 0, & \text{for } l = 0 \\ 2k = 2l', & \text{for even } l \neq 0 \\ 2k - 1 = 2(k + N') = 2l', & \text{for odd } l, \end{cases} \quad (29)$$

where $N' = (N - 1)/2$, $k = 1, 2, \dots, N'$, and $l' = 1, 2, \dots, 2N' = N - 1$. In this way a relation similar to (14) is obtained

$${}_1^l D_{jk}[X, -Y] = {}_2^{l'} D_{jk} \left[\tau([X, Y] | [X, Y]) \right], \quad (30)$$

where l and l' are interrelated by (29). In the same way one can transform the factor system (15) into (9). If $\gcd(l, N) = 1$ then l is replaced by l' , so

$$m_l([X, Y], [X', Y']) = \exp(2\pi i (2l')YX'/N) \quad (31)$$

and next w is taken to be l'/N . The factor system obtained

$$t_w m_l([X, Y], [X', Y']) = \exp[\pi i l(YX' - XY')/N] \quad (32)$$

is exactly the same as (9). In a sense, we have performed a ‘normalization’ of the factor system used by Brown and Zak. In other words, the projective representations (10) do not satisfy the condition (24) for odd N since they are not given in *normalized* form. Such a form can be obtained by substitution $l \rightarrow 2l'$ which leads to the condition (12) determined by Zak.

Anyway, this way of normalization is not possible in the case of even N , since in general $\gcd(l, N) \neq \gcd(2l, N)$. However, we can use a hint given by Zak, who did not exploit it in full. At the end of his paper⁹ Zak noticed that a finite magnetic translation group contains N^3 elements²¹ for odd N whereas for even N the number of elements is two times bigger. It suggests that a group considered by him was, in fact, an extension of T_N by C_{2N} — the factors obtained were not normalized since they did not belong to the multiplier of T_N .

In a previous paper¹⁵ it was shown that central extensions of T_N by C_{2N} which correspond to magnetic translation groups have factor systems

$${}_2 m_k([X, Y], [X', Y']) = \exp\left(\frac{\pi i}{N} 2kYX'\right) \quad (33)$$

with $k = 0, 1, \dots, N - 1$. A mapping f_w assigns to each $[X, Y]$ an element of C_{2N} so it is well defined for $w = k/2N$, *i.e.*,

$$f_w[X, Y] = \exp\left(\frac{\pi i}{N} kXY\right).$$

Note that the product kXY is calculated modulo $2N$ and, therefore, f_w is a multivalued function: in \mathbb{Z}_N numbers X and $X + N$ represent the same element, whereas

$$f_w[X + N, Y] = (-1)^{kY} \exp\left(\frac{\pi i}{N} kXY\right)$$

is not equal to $f_w[X, Y]$, in a general case. To calculate a trivial factor system t_w according to (19) one has to determine $f_w(\mathbf{R} + \mathbf{R}')$. Let us assume, at this moment only formally, that a sum of vectors in this formula will *not* be calculated modulo N . Then

$$t_w([X, Y], [X', Y']) = \exp\left[-\frac{\pi i}{N}k(XY' + X'Y)\right] \quad (34)$$

and

$${}_2m_k t_w([X, Y], [X', Y']) = \exp\left[\frac{\pi i}{N}k(YX' - XY')\right],$$

which coincides with (9). It means that in order to obtain the above results we have to treat $[X, Y]$ as an element of T_{2N} rather than that of T_N . So, in fact, we have considered a larger, $2N \times 2N$, lattice although a parameter labeling nonequivalent central extensions has been taken to be equal $2k$. Even for $\gcd(k, N) = 1$ we have $\gcd(2k, 2N) = 2$, so the condition accompanying (8) that l is mutually prime with the period, is not fulfilled in this case. This leads to the magnetic periodicity with a period N though the lattice (crystal) period is $2N$. This problem was briefly discussed by Brown⁸ and Zak¹⁰ and its solution is possible through a concept of *magnetic cells* — if $\gcd(l, N) = \lambda > 1$ [in (8) and (12)] then the $N \times N$ lattice can be decomposed into $(N/\lambda) \times (N/\lambda)$ magnetically periodic sublattices, which form a $\lambda \times \lambda$ lattice of magnetic cells. In the case considered $\lambda = 2$ and the $(2N) \times (2N)$ lattice is decomposed into four $N \times N$ sublattices consisting of points $[X, Y]$, $[X + N, Y]$, $[X, Y + N]$, and $[X + N, Y + N]$, respectively, where $X, Y \in \mathbb{Z}_N$. According to (17) and (33) an irreducible projective representation of T_{2N} should be N -dimensional in this case. It follows from the Burnside's theorem that there are *four* such representations and they can be chosen as

$${}_3D_{jk}^{\kappa_x, \kappa_y}[X, Y] = (-1)^{\kappa_x \epsilon_x + \kappa_y \epsilon_y} \exp\left(2\pi i \frac{l}{N} Xj\right) \delta_{j, k-Y}, \quad (35)$$

where $\kappa_x, \kappa_y = 0, 1$ and ϵ_x (ϵ_y) is equal to 0 for $X < N$ ($Y < N$) and to 1 otherwise. Therefore the representations considered by Brown and Zak (14) are equivalent to ${}_3D^{0,0}$. However, the latter is clearly periodic with the period N and satisfies the condition (25). Since the trivial factor system (34) is not normalized in C_N then the representations (14) satisfy the condition (24) and do not satisfy (25).

V. DISCUSSION AND FINAL REMARKS

Summarizing the above discussion on odd and even periods N we can state that the antisymmetric gauge $\mathbf{A} = [Hy, -Hx]/2$, considered by Brown and Zak, corresponds in fact to the crystal period $2N$ and the magnetic period N . If one, like Brown and Zak, does not take this fact into account then results obtained can lead to erroneous conclusions. For example, in this way the additional condition (12) was derived. Investigations of the standard and normalized factor system (15), corresponding to the Landau gauge, have clearly indicated points at which the magnetic translation groups are 'gauge-dependent' and how one can 'normalize' factor systems and representations.

The magnetically periodic boundary conditions of projective representations, when the Landau gauge is assumed, can be invoked if

$$H = \frac{l}{N} \varphi_0, \quad (36)$$

where $l = 0, 1, \dots, N-1$. (If $\gcd(l, N) = \lambda$ then the magnetic period is equal to N/λ , whereas the crystal period is still N .) The factor system (15) (and also the representation (17) and the physical properties) is a periodic function of H with a period φ_0 . Hence the different magnetic response of the considered system can be observed only for N values of $H = l\varphi_0/N$.

If N is an odd integer then $\gcd(2l, N) = \gcd(l, N)$ so the magnetic periodicity is the same in both cases and l in (36) can be replaced by $2l'$, which is equivalent to Zak's condition (12). However, the successively counted values of HN/φ_0 have to be arranged in a different order: $0, N+1, 2, N+3, \dots, 2N-2, N-1$. If these values were arranged in the increasing order, *i.e.*, $0, 2, \dots, N-1, N+1, \dots, 2N-2$, it might suggest that the condition (12) has to be taken into account and that the magnetic period is $2\varphi_0$. The only way to settle this problem is by investigation of a system described by a Hamiltonian with a nonperiodic part, *e.g.*, the paramagnetic term.

The case of even N has a quite different nature. As was shown above, the factor systems and representations considered by Brown and Zak describe a lattice with the crystal period $2N$ and the magnetic period equal N . The condition (36) yields $H = l\varphi_0/(2N)$, with $l = 0, 1, \dots, 2N-1$, but to achieve the magnetic period N only even values of $l = 2l'$ are considered, so $H = l'\varphi_0/N$. The representation (10) [see also (14)] is one of four nonequivalent irreducible representations which can be determined in this case. It can be easily seen that Brown's considerations for odd N can also be interpreted in this way (since the decomposition of a $(2N) \times (2N)$ lattice into four $N \times N$

lattices does not depend on the parity of N). It means, in particular, that the antisymmetric gauge for $N = 2$ can be introduced only if one considers the 4×4 lattice with $H = 0$ (a trivial case) or $H = \varphi_0/2$.

In this work the descriptions of the magnetic translation group proposed by Brown and Zak were compared with the results obtained by means of the Mac Lane method.¹⁵ The first authors assumed the antisymmetric gauge, whereas the Mac Lane method led to the Landau gauge. Due to a factor $\frac{1}{2}$ in the antisymmetric gauge some problems arise when one introduces the magnetically periodic boundary conditions. More careful considerations put forward by Zak gave the additional condition (12) for an odd period N . However, the condition (36) obtained for the Landau gauge resembles the Brown's condition (8) and does not depend on the parity of N . This condition was obtained from the group-theoretical considerations leading to the factor system (15). In the next step k/N was interpreted as H/φ_0 . At first sight it can be interpreted as any value proportional to H , *e.g.*, as $2H/\varphi_0$. However, the first choice is confirmed by the value of group-theoretical commutator, which does not depend on the gauge or, in the other words, is identical for all equivalent factor systems.

Let us also remind that in this work H in fact denotes the magnetic flux through one primitive cell. Therefore, according to (8) or (36), the total flux through the $N \times N$ lattice is equal to

$$\Phi = lN\varphi_0, \quad (37)$$

i.e., to an integer multiplicity of the fluxon. To introduce the antisymmetric gauge one has to consider a $(2N) \times (2N)$ lattice and even $l = 2l'$. Hence the total flux equals $\Phi = 4l'N\varphi_0$, so the flux through one $N \times N$ magnetic cell is equal to $l'N\varphi_0$, which is consistent with the previous value (37), and the flux through one primitive cell is equal to $H_a = l'\varphi_0/N$. On the other hand, the flux through one primitive cell of the $(2N) \times (2N)$ lattice (assuming the Landau gauge) is $H_L = l\varphi_0/(2N)$, so in general it is a half of H_a . Therefore we can set up the certain procedure: For a given magnetic field H_a the antisymmetric gauge can be introduced if the magnetically periodic boundary conditions admit two times smaller H_L . For the sake of illustration let us consider the $(2N) \times (2N)$ lattice and $H = \varphi_0/N$. Then, from (16), one obtains a commutator corresponding to the primitive vectors $[1, 0]$ and $[0, 1]$ as

$$c([1, 0], [0, 1]) = \exp(-2\pi i/N).$$

The formula (9) gives the following values of the corresponding factors [see (26)]

$${}_1m([1, 0], [0, 1]) = \exp(-\pi i/N)$$

and

$${}_1m([0, 1], [1, 0]) = \exp(\pi i/N),$$

whereas (15) leads to

$${}_2m([1, 0], [0, 1]) = 1$$

and

$${}_2m([0, 1], [1, 0]) = \exp(2\pi i/N).$$

So, the flux through the primitive cell, corresponding to the commutator and independent of the gauge, was decomposed in two different ways into fluxes through 'primitive' triangles. However, the first decomposition (related to the antisymmetric gauge) is not possible for the minimal flux $H = \varphi_0/(2N)$. If considering any other trivial factor system (27) determined by the parameter $w \in \mathbb{R}$ one can obtain many other decompositions of the commutator into factors. It can be viewed as the decomposition of the flux through the primitive cell into fluxes through the 'lower' and 'upper' triangle. In particular, the other Landau gauge, corresponding to the factor system (28), changes roles of these triangles since one obtains

$$\overline{m}([1, 0], [0, 1]) = \exp(2\pi i/N)$$

and

$$\overline{m}([0, 1], [1, 0]) = 1.$$

Despite the fact that the physical properties are gauge-independent we have noticed that the form of the vector potential \mathbf{A} has a certain importance in the mathematical description of a system. One has to be especially very careful considering projective representations or extensions of groups, since some equivalent factor systems are neither standard nor normalized. However, it may occur that in certain applications or in other descriptions of the same problem it is more convenient to use such a form of \mathbf{A} .

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